VectorBiTE Methods Training Bayesian State Space Modeling for Time Series Data

The VectorBiTE Team
(John W. Smith, Virginia Tech)

Summer 2021

Prerequisites

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- Rough understanding of Bayesian statistics and Likelihoods

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- Rough understanding of Bayesian statistics and Likelihoods
- Exposure to `JAGS` and related MCMC packages

Why State Space Models?

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- The models that we use for physical processes are not perfect

Rogue waves, long thought to be sailor folklore, have been verified and have caused scientists to reconsider their theory behind waves

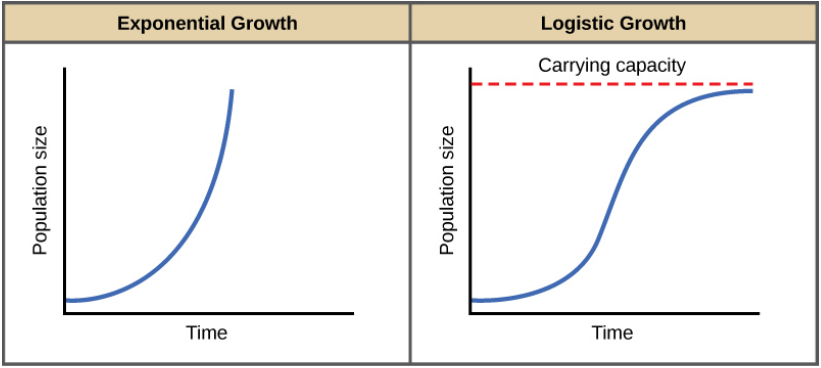
Why State Space Models?

- Ability to make inference on unobserved processes

Temperature is a common variable used as a proxy for Mosquito life traits

Why State Space Models?

- Physical processes are often non-linear and/or non-Gaussian



Population growth is an example of a physical process that is typically non-linear and/or non-Gaussian

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- $y_{1:T}$ are independent of one another conditional on $x_{1:T}$

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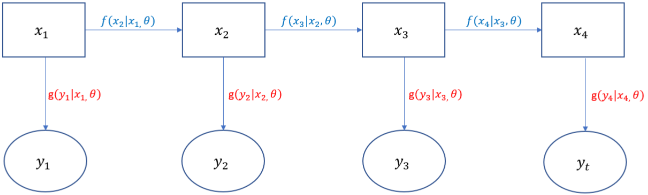
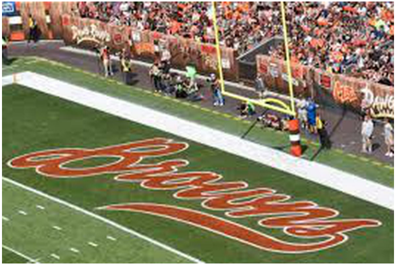
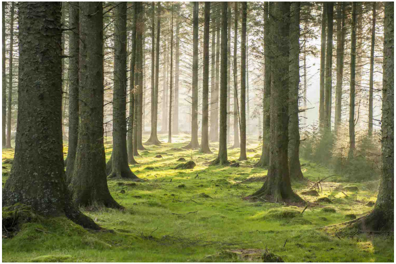
- $f(\cdot)$ and $g(\cdot)$ are both stochastic

Illustration



What Can We Use SSMs For?





State Space Models: Example 1

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Let's consider the following example, and identify the corresponding $\Theta, f(x_t|x_{t-1}, \Theta)$, and $g(y_t|x_t, \Theta)$.

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- This is a special case of the State Space Model called a *Linear Gaussian State Space model*, or *Normal Dynamic Linear model* (NDLM)

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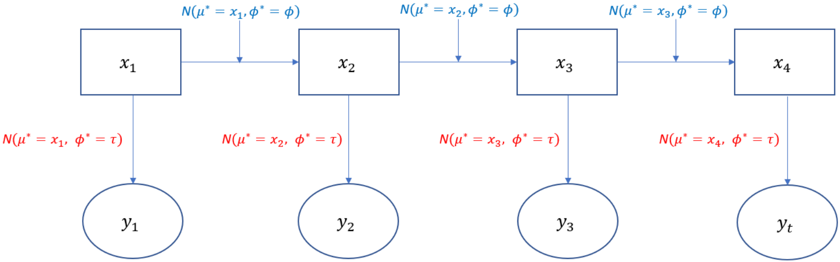

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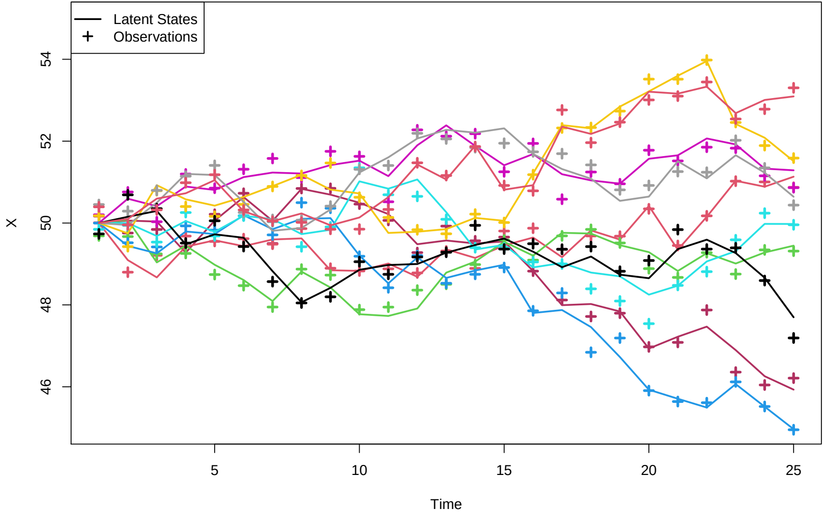

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$$
\begin{aligned}
x_t &= \frac{x_{t-1}}{2} + 25\frac{x_{t-1}}{1+x_{t-1}^2} + 8\cos(1.2t) + \epsilon_{proc} \\
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- $g(y_t|x_t, \Theta) \sim N(\mu^* = \frac{x_t^2}{20}, \phi^* = \tau)$
- This is an example of a *Gaussian State Space model*. It differs from Example 1 because it is not linear (why?)

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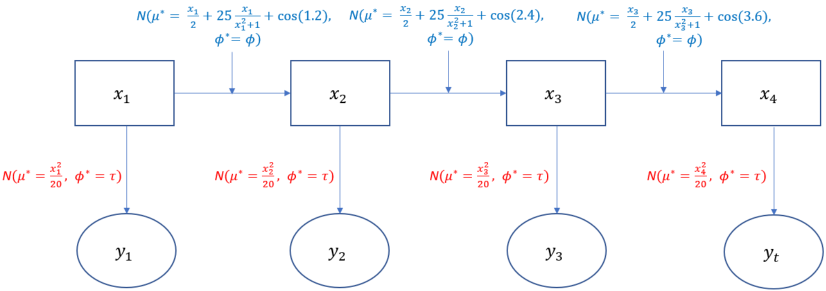

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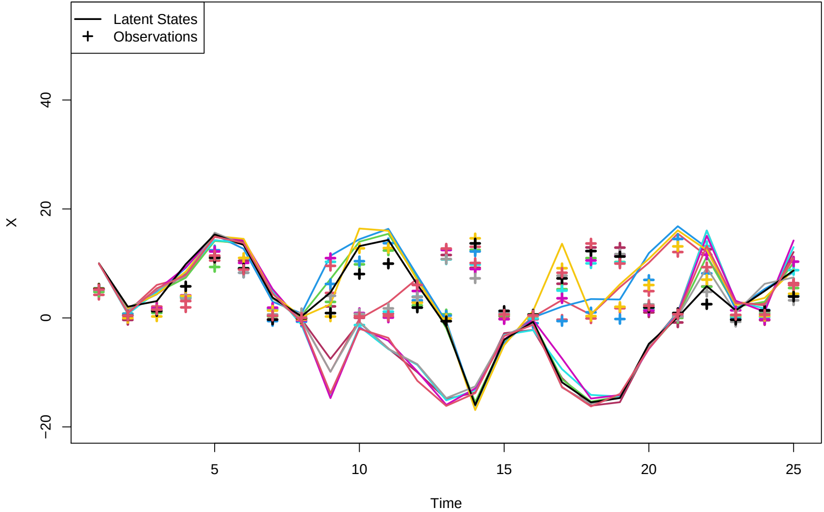

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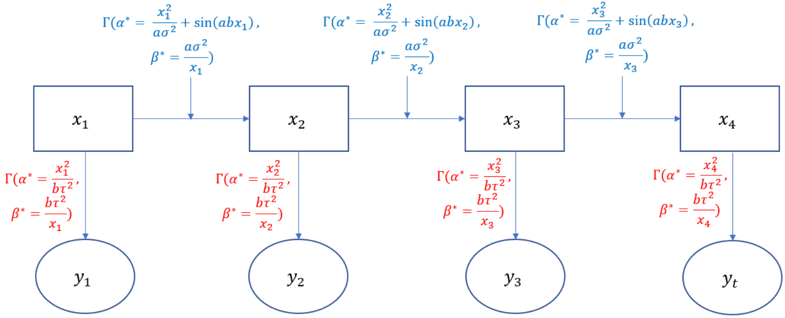

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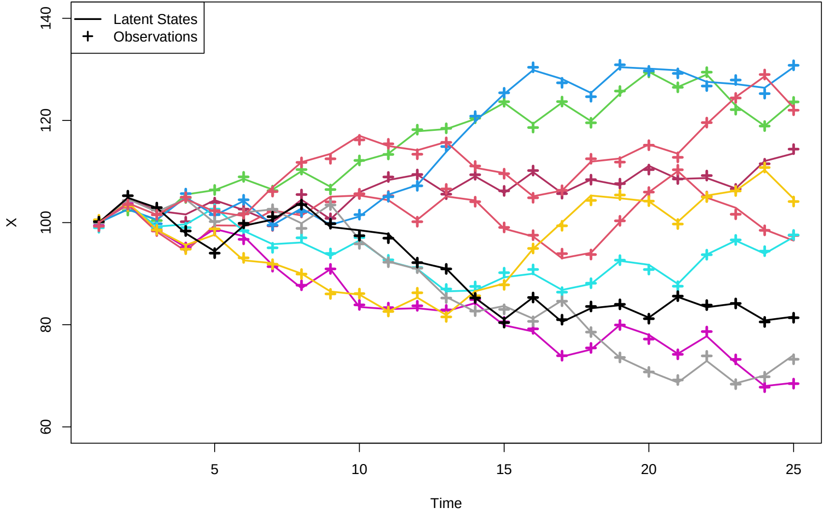

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  - Though not all parameters in Θ need to be known, we need to know the form of the distribution for the evolution and observation density functions

End of Presentation 1